

On optimal approximability results for computing the strong metric dimension*

Bhaskar DasGupta and Nasim Mobasher
 Department of Computer Science
 University of Illinois at Chicago
 Chicago, IL 60607, USA
 Emails: {bdasgup,nmobas2}@uic.edu

August 7, 2014

Abstract

In this short note, we observe that the problem of computing the strong metric dimension of a graph can be reduced to the problem of computing a minimum node cover of a transformed graph within an *additive* logarithmic factor. This implies both a 2-approximation algorithm and a $(2 - \varepsilon)$ -inapproximability for the problem of computing the strong metric dimension of a graph.

1 Introduction

The strong metric dimension of a graph was introduced in [7] as an alternative to the previously introduced (weak) metric dimension of graphs [2, 8]. Subsequently, the strong metric dimension has been investigated in several research papers such as [5, 6, 10]. Let $G = (V, E)$ be a given undirected graph of n nodes. To define the strong metric dimension, we will use the following notations and terminologies:

- $\text{Nbr}(u) = \{v \mid \{u, v\} \in E\}$ is the set of neighbors of (*i.e.*, nodes adjacent to) a node u .
- $u \overset{s}{\rightsquigarrow} v$ denotes a shortest path from between nodes u and v of length (number of edges) $d_{u,v} = \ell(u \overset{s}{\rightsquigarrow} v)$.
- $\text{diam}(G) = \max_{u,v \in V} \{d_{u,v}\}$ denotes the diameter of a graph G .
- A shortest path $u \overset{s}{\rightsquigarrow} v$ is *maximal* if and only if it is not properly included inside another shortest path, *i.e.*, if and only if

$$(\forall x \in \text{Nbr}(u) \ d(x, v) \leq d(u, v)) \wedge (\forall y \in \text{Nbr}(v) \ d(y, u) \leq d(u, v))$$

- A node x *strongly resolves* a pair of nodes u and v , denoted by $x \blacktriangleright \{u, v\}$, if and only if either v is on a shortest path between x and u or either u is on a shortest path between x and v .
- A set of nodes $V' \subseteq V$ is a *strongly resolving set* for G , denoted by $V' \blacktriangleright G$, if and only if every distinct pair of nodes of G is strongly resolved by some node in V' , *i.e.*, if and only if

$$\forall (u, v \in V, u \neq v) \exists x \in V' : x \blacktriangleright \{u, v\}$$

Then, the problem of computing the string metric dimension of a graph is defined as shown below:

*Research partially supported by NSF grants IIS-1160995.

Problem name:	Strong Metric Dimension (STR-MET-DIM)
Instance:	an undirected graph $G = (V, E)$.
Valid Solution:	a set of nodes $V' \subseteq V$ such that $V' \blacktriangleright G$.
Goal:	<i>minimize</i> $ V' $.
Related notation:	$\text{sdim}(G) = \min_{V' \subseteq V \wedge V' \blacktriangleright G} \{ V' \}$.

In this short note, we observe that the problem of computing the strong metric dimension of a graph can be reduced to the problem of computing a minimum node cover of a transformed graph within an *additive* logarithmic factor. This implies both a 2-approximation algorithm and a $(2 - \varepsilon)$ -inapproximability for the problem of computing the strong metric dimension of a graph. More precisely, our result is summarized by the following Lemma.

Lemma 1.1.

(a) STR-MET-DIM admits a polynomial-time 2-approximation.

(b) Assuming the unique games conjecture¹ (UGC) is true, STR-MET-DIM does not admit any polynomial-time $(2 - \varepsilon)$ -approximation for any constant $\varepsilon > 0$ even if the given graph is restricted in the sense that

- (i) $\text{diam}(G) \leq 2$, or
- (ii) G is bipartite and $\text{diam}(G) \leq 4$.

Remark 1.2. If instead of assuming the correctness of UGC the standard assumption of $P \neq \text{NP}$ is made, then the part (b) of the above theorem still holds provided one replaces $(2 - \varepsilon)$ -inapproximability by 1.36-inapproximability. This is easily obtained by a similar proof in which we use the $(10\sqrt{5} - 21 \approx 1.3606)$ -inapproximability result.

2 Proof of Theorem 1.1

The standard minimum node cover (MNC) problem for a graph is defined as follows:

Instance:	an undirected graph $G = (V, E)$.
Valid Solution:	a set of nodes $V' \subseteq V$ such that $V' \cap \{u, v\} \neq \emptyset$ for every edge $\{u, v\} \in E$.
Goal:	<i>minimize</i> $ V' $.
Related notation:	$\text{MNC}(G) = \min_{\forall \{u, v\} \in E : V' \cap \{u, v\} \neq \emptyset} \{ V' \}$.

Let $G = (V, E)$ denote the input graph of n nodes. We recall the following result from [5].

Theorem 2.1. [5]

(a) Let $\widehat{G} = (V, \widehat{E})$ be the graph obtained from G in the following manner:

$$\{u, v\} \in \widehat{E} \equiv u \neq v \text{ and } u \overset{s}{\longleftrightarrow} v \text{ is a maximal shortest path in } G$$

Then $\text{sdim}(G) = \text{MNC}(\widehat{G})$ and $V' \subseteq V$ is a valid solution of STR-MET-DIM on G if and only if V' is a valid solution of MNC on \widehat{G} .

¹See [3] for a definition of the unique games conjecture.

(b) Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the graph from G built in the following manner:

- Let $u_1, u_2, \dots, u_\kappa$ be the nodes in G such that, for every u_i ($1 \leq i \leq \kappa$), there is a node $v_i \neq u_i$ in G with the property that $\text{Nbr}(u_i) = \text{Nbr}(v_i)$.
- Let $\overline{G} = (V, \overline{E})$ be the (edge) of G , i.e., $\{u, v\} \in \overline{E} \equiv \{u, v\} \notin E$. Then \tilde{G} is constructed as follows:
 - $\tilde{V} = V \cup \{x_1, x_2, \dots, x_k, y\}$ where $x_1, x_2, \dots, x_k, y \notin V$.
 - $\tilde{E} = \overline{E} \cup \left(\bigcup_{j=1}^k \{x_j, u_j\} \right) \cup \left(\bigcup_{y' \in \tilde{V} \setminus \{y\}} \{y', y\} \right)$.

Then, $\text{diam}(\tilde{G}) = 2$ and $\text{sdim}(\tilde{G}) = \kappa + \text{MNC}(G)$.

Proof of Lemma 1.1(a)

Follows from Fact 2.1(a) and a well-known 2-approximation algorithm for MNC [9, Theorem 1.3].

Proof of Lemma 1.1(b)

Consider the standard Boolean satisfiability problem (SAT) and let Φ be an input instance of SAT. Our starting point is the following inapproximability result proved Khot and Regev [4]:

[4, setting $k = 2$] Assuming UGC is true, there exists a polynomial time algorithm that transforms a given instance Φ of SAT to an input instance graph $G = (V, E)$ of MNC with n nodes such that, for any arbitrarily small constant $\varepsilon > 0$, the following holds:

- (★) **(YES case)** if Φ is satisfiable then $\text{MNC}(G) \leq \left(\frac{1}{2} + \varepsilon\right)n$, and
 (NO case) if Φ is not satisfiable then $\text{MNC}(G) \geq (1 - \varepsilon)n$.

Consider such an instance G of MNC as generated by the above transformation. We first construct the following graph $G^+ = (V^+, E^+)$ from G . Let $k = 1 + \lfloor \log_2 n \rfloor$ and let $b(j) = b_{k-1}(j) b_{k-2}(j) \dots b_1(j) b_0(j)$ be the binary representation of an integer $j \in \{1, 2, \dots, n\}$ using exactly k bits (e.g., if $n = 5$ then $b(3) = \overset{b_2(3)}{0} \overset{b_1(3)}{1} \overset{b_0(3)}{1}$). Let u_1, u_2, \dots, u_n be an arbitrary ordering of the nodes in V . Then,

- $V^+ = V \cup V_1^+$ where $V_1^+ = \{v_1, v_2, \dots, v_{k-1}, y\}$ is a set of k new nodes, and
- $E^+ = E \cup \left(\bigcup_{j=1}^n \{ \{u_j, v_\ell\} \mid b_\ell(j) = 1 \} \right) \cup \left(\bigcup_{j=1}^{k-1} \{y, v_j\} \right)$.

Thus $|V^+| = n + k$ and $|E^+| < |E| + \frac{n^k}{2} + k$. Now, note that if $V' \subseteq V$ is a solution of MNC on G , then $V' \cup V_1^+$ is a solution of MNC on G^+ , implying $\text{MNC}(G^+) \leq \text{MNC}(G) + k$, and, conversely, if $V' \subseteq V^+$ is a solution of MNC on G^+ , then $V' \setminus V_1^+$ is a solution of MNC on G , implying $\text{MNC}(G) \leq \text{MNC}(G^+)$. Combining the above inequalities with that in (★), we have

- (★★) **(YES case)** if Φ is satisfiable then $\text{MNC}(G^+) < \left(\frac{1}{2} + \varepsilon\right)n + \log_2 n + 1$, and
 (NO case) if Φ is not satisfiable then $\text{MNC}(G^+) \geq (1 - \varepsilon)n$.

We now build the graph $\tilde{G}^+ = (\tilde{V}^+, \tilde{E}^+)$ from G using the construction in Fact 2.1 and observe the following:

- For any $i \neq j$, since $b(i) \neq b(j)$, there exists an index t such that $b_t(i) \neq b_t(j)$, say $b_t(i) = 0$ and $b_t(j) = 1$. Thus, $\text{Nbr}(u_i) \neq \text{Nbr}(u_j)$ since $v_t \in \text{Nbr}(u_j)$ but $v_t \notin \text{Nbr}(u_i)$.
- Since $b(i) \neq 0$ for any i and $b(1), b(2), \dots, b(n)$ are distinct binary numbers each of exactly k bits, for any $t \neq t'$ there is an index i such that $b_t(i) \neq b_{t'}(i)$, say $b_t(i) = 0$ and $b_{t'}(i) = 1$. Thus, $\text{Nbr}(v_t) \neq \text{Nbr}(v_{t'})$ since $u_i \in \text{Nbr}(v_{t'})$ but $u_i \notin \text{Nbr}(v_t)$.

- For any i and j , $\text{Nbr}(u_i) \neq \text{Nbr}(v_j)$ since $y \in \text{Nbr}(v_j)$ but $y \notin \text{Nbr}(u_i)$.
- For any i , $b(i) \neq 0$ and thus there exists an index j such that $b_j(i) = 1$. This implies $u_j \in \text{Nbr}(v_i)$ but $u_j \notin \text{Nbr}(y)$ and therefore $\text{Nbr}(v_i) \neq \text{Nbr}(y)$.
- Since G is a connected graph, for every node u_i there exists a node u_j such that $\{u_i, u_j\} \in E^+$. Thus, $u_j \in \text{Nbr}(u_i)$ but $u_j \notin \text{Nbr}(y)$, implying $\text{Nbr}(u_i) \neq \text{Nbr}(y)$.

Thus, no two nodes in G^+ have the same neighborhood, implying $\kappa = 0$ and $\text{sdim}(\widehat{G}^+) = \text{MNC}(G^+)$. Thus, setting $\varepsilon' = \varepsilon + \frac{\log_2 n + 1}{n} > \varepsilon$ to be any arbitrarily small constant, it follows from $(\star\star)$ that

- ($\star\star\star$) **(YES case)** if Φ is satisfiable then $\text{MNC}(G^+) < (\frac{1}{2} + \varepsilon')n$, and
 (NO case) if Φ is not satisfiable then $\text{MNC}(G^+) \geq (1 - \varepsilon')n$.

This proves the claim in **(b)(i)**. To prove **(b)(ii)**, we modify the graph \widehat{G}^+ to a new graph $G' = (V', E')$ by splitting every edge into a sequence of two edges, i.e., for every edge $\{u, v\}$ in \widehat{G}^+ we add a new node x_{uv} in G' and replace the edge $\{u, v\}$ by the two edges $\{u, x_{uv}\}$ and $\{v, x_{uv}\}$. Clearly G' is bipartite since all its cycles are of even length and $\text{diam}(G') \leq 2 + \text{diam}(\widehat{G}^+) = 4$. To show that $\text{sdim}(\widehat{G}^+) = \text{sdim}(G')$, by Fact 2.1(a) it suffices to show that no maximal shortest path ends at a node x_{uv} . Indeed, if a maximal shortest path \mathcal{P} from some node z ends at x_{uv} , it must use one of the two edges $\{u, x_{uv}\}$ and $\{v, x_{uv}\}$, say $\{u, x_{uv}\}$. Then adding the edge $\{v, x_{uv}\}$ to the path provide a shortest path between v and z and thus \mathcal{P} was not maximal. As a result, the inapproximability result for \widehat{G}^+ directly translates to that for G' .

References

- [1] I. Dinur, S. Safra. *On the hardness of approximating minimum vertex cover*, Annals of Mathematics, 162(1), 439-485, 2005.
- [2] F. Harary and R. A. Melter. *On the metric dimension of a graph*, Ars Combinatoria, 2, 191-195, 1976.
- [3] S. Khot. *On the power of unique 2-Prover 1-Round games*, 34th ACM Symposium on Theory of Computing, 767-775, 2002.
- [4] S. Khot and O. Regev. *Vertex cover might be hard to approximate to within $2-\varepsilon$* , Journal of Computer and System Sciences, 74(3), 335-349, 2008.
- [5] O. R. Oellermann and J. Peters-Fransen. *The strong metric dimension of graphs and digraphs*, Discrete Applied Mathematics, 155, 356-364, 2007.
- [6] J. A. Rodríguez-Velázquez, I. G. Yerob, D. Kuziaka and O. R. Oellermann. *On the strong metric dimension of Cartesian and direct products of graphs*, Discrete Mathematics, 335, 8-19, 2014.
- [7] A. Sebø and E. Tannier. *On Metric Generators of Graphs*, Mathematics of Operations Research, 29(2), 383-393, 2004.
- [8] P. J. Slater. *Leaves of trees*, Congressus Numerantium, 14, 549-559, 1975.
- [9] V. Vazirani. *Approximation Algorithms*, Springer-Verlag, 2001.
- [10] E. Yi. *On Strong Metric Dimension of Graphs and Their Complements*, Acta Mathematica Sinica, English Series, 29(8), 1479-1492, 2013.